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## LETTER TO THE EDITOR

**Structure constants for new infinite-dimensional Lie algebras of  $U(N_+, N_-)$  tensor operators and applications**

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**Abstract.** The structure constants for Moyal brackets of an infinite basis of functions on the algebraic manifolds  $M$  of pseudo-unitary groups  $U(N_+, N_-)$  are provided. They generalize the Virasoro and  $\mathcal{W}_\infty$  algebras to higher dimensions. The connection with volume-preserving diffeomorphisms on  $M$ , higher generalized-spin and tensor operator algebras of  $U(N_+, N_-)$  is discussed. These centrally extended, infinite-dimensional Lie algebras also provide the arena for nonlinear integrable field theories in higher dimensions, residual gauge symmetries of higher-extended objects in the light-cone gauge and  $C^*$ -algebras for tractable non-commutative versions of symmetric curved spaces.

The general study of infinite-dimensional algebras and groups, their quantum deformations (in particular, central extensions) and representation theory has not progressed very far, except for some important achievements in one- and two-dimensional systems, and there can be no doubt that a breakthrough in the subject would provide new insights into the two central problems of modern physics: unification of all interactions and exact solvability in QFT and statistics.

The aforementioned achievements refer mainly to Virasoro and Kac–Moody symmetries (see, e.g., [1, 2]), which have played a fundamental role in the analysis and formulation of conformally invariant (quantum and statistical) field theories in one and two dimensions, and systems in higher dimensions which in some essential respects are one- or two-dimensional (e.g. string theory). Generalizations of the Virasoro symmetry, as the algebra  $\text{diff}(S^1)$  of reparametrizations of the circle, led to the infinite-dimensional Lie algebras of area-preserving diffeomorphisms  $\text{sdiff}(\Sigma)$  of two-dimensional surfaces  $\Sigma$ . These algebras naturally appear as a residual gauge symmetry in the theory of relativistic membranes [3], which exhibits an intriguing connection with the quantum mechanics of space constant (e.g. vacuum configurations)  $SU(N)$  Yang–Mills potentials in the limit  $N \rightarrow \infty$  [4]; the argument that the internal symmetry space of the  $U(\infty)$  pure Yang–Mills theory must be a functional space, actually the space of configurations of a string, was pointed out in [5]. Moreover, the  $\mathcal{W}_\infty$  and  $\mathcal{W}_{1+\infty}$  algebras of area-preserving diffeomorphisms of the cylinder [6] generalize the underlying Virasoro gauged symmetry of the light-cone two-dimensional induced gravity discovered by Polyakov [7] by including all positive conformal-spin currents [8], and induced actions for these  $\mathcal{W}$ -gravity theories have been proposed [9, 10]. Additionally, the  $\mathcal{W}_{1+\infty}$  (dynamical) symmetry has been identified by the authors of [11] as the set of canonical

transformations that leave invariant the Hamiltonian of a two-dimensional electron gas in a perpendicular magnetic field, and appears to be relevant in the classification of all the universality classes of *incompressible quantum fluids* and the identification of the quantum numbers of the excitations in the quantum Hall effect. Higher-spin symmetry algebras were introduced in [12] and could provide a guiding principle towards the still unknown ‘M-theory’.

It is remarkable that area-preserving diffeomorphisms, higher-spin and  $\mathcal{W}$  algebras can be seen as distinct members of a one-parameter family  $\mathcal{L}_\mu(su(2))$ —or the non-compact version  $\mathcal{L}_\mu(su(1, 1))$ —of non-isomorphic [13] infinite-dimensional Lie algebras of  $SU(2)$ —and  $SU(1, 1)$ —tensor operators, more precisely, the factor algebra  $\mathcal{L}_\mu(su(2)) = \mathcal{U}(su(2))/\mathcal{I}_\mu$  of the universal enveloping algebra  $\mathcal{U}(su(2))$  by the ideal  $\mathcal{I}_\mu = (\hat{C} - \hbar^2\mu)\mathcal{U}(su(2))$  generated by the Casimir operator  $\hat{C}$  of  $su(2)$  ( $\mu$  denotes an arbitrary complex number). The structure constants for  $\mathcal{L}_\mu(su(2))$  and  $\mathcal{L}_\mu(su(1, 1))$  are well known for the Racah–Wigner basis of tensor operators [14], and they can be written in terms of Clebsch–Gordan and (generalized)  $6j$ -symbols [3, 8, 15]. Another interesting feature of  $\mathcal{L}_\mu(su(2))$  is that, when  $\mu$  coincides with the eigenvalue of  $\hat{C}$  in an irrep  $D_j$  of  $SU(2)$ , that is  $\mu = j(j+1)$ , there exists an ideal  $\chi$  in  $\mathcal{L}_\mu(su(2))$  such that the quotient  $\mathcal{L}_\mu(su(2))/\chi \simeq sl(2j+1, C)$  or  $su(2j+1)$ , by taking a compact real form of the complex Lie algebra [16]. That is, for  $\mu = j(j+1)$  the infinite-dimensional algebra  $\mathcal{L}_\mu(su(2))$  collapses to a finite-dimensional one. This fact was used in [3] to approximate  $\lim_{\hbar \rightarrow 0} \lim_{\mu \rightarrow \infty} \mathcal{L}_\mu(su(2)) \simeq \text{sdiff}(S^2)$  by  $su(N)|_{N \rightarrow \infty}$  (‘large number of colours’).

The generalization of these constructions to general unitary groups proves to be quite unwieldy, and a canonical classification of  $U(N)$ -tensor operators has, so far, been proven to exist only for  $U(2)$  and  $U(3)$  (see [14] and references therein). Tensor labelling is provided in these cases by the Gel’fand–Weyl pattern for vectors in the carrier space of the irreps of  $U(N)$ .

In this letter, a quite appropriate basis of operators for  $\mathcal{L}_{\vec{\mu}}(u(N_+, N_-))$ ,  $\vec{\mu} = (\mu_1, \dots, \mu_N)$ ,  $N \equiv N_+ + N_-$ , is provided and the structure constants, for the particular case of the boson realization of quantum associative operatorial algebras on algebraic manifolds  $M_{N_+, N_-} = U(N_+, N_-)/U(1)^N$ , are calculated. The particular set of operators in  $\mathcal{U}(u(N_+, N_-))$  is the following:

$$\begin{aligned} \hat{L}_{|m|}^I &\equiv \prod_{\alpha} (\hat{G}_{\alpha\alpha})^{I_{\alpha} - (\sum_{\beta > \alpha} |m_{\alpha\beta}| + \sum_{\beta < \alpha} |m_{\beta\alpha}|)/2} \prod_{\alpha < \beta} (\hat{G}_{\alpha\beta})^{|m_{\alpha\beta}|} \\ \hat{L}_{-|m|}^I &\equiv \prod_{\alpha} (\hat{G}_{\alpha\alpha})^{I_{\alpha} - (\sum_{\beta > \alpha} |m_{\alpha\beta}| + \sum_{\beta < \alpha} |m_{\beta\alpha}|)/2} \prod_{\alpha < \beta} (\hat{G}_{\beta\alpha})^{|m_{\alpha\beta}|} \end{aligned} \quad (1)$$

where  $\hat{G}_{\alpha\beta}$ ,  $\alpha, \beta = 1, \dots, N$ , are the  $U(N_+, N_-)$  Lie-algebra (step) generators with commutation relations

$$[\hat{G}_{\alpha_1\beta_1}, \hat{G}_{\alpha_2\beta_2}] = \hbar(\eta_{\alpha_1\beta_2} \hat{G}_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1} \hat{G}_{\alpha_1\beta_2}) \quad (2)$$

and  $\eta = \text{diag}(1, \dots, 1, -1, \dots, -1)$  is used to raise and lower indices; the upper (generalized spin) index  $I \equiv (I_1, \dots, I_N)$  of  $\hat{L}$  in (1) represents an  $N$ -dimensional vector which, for the present, is taken to lie on an half-integral lattice; the lower index  $m$  symbolizes a integral upper-triangular  $N \times N$  matrix, and  $|m|$  denotes the absolute value of all its entries. Thus, the operators  $\hat{L}_m^I$  are labelled by  $N + N(N-1)/2 = N(N+1)/2$  indices, in the same way as wavefunctions  $\psi_m^I$  in the carrier space of irreps of  $U(N)$ . An implicit quotient by the ideal  $\mathcal{I}_{\vec{\mu}} = \prod_{j=1}^N (\hat{C}_j - \hbar^j \mu_j)\mathcal{U}(u(N_+, N_-))$  generated by the Casimir operators

$$\hat{C}_1 = \hat{G}_{\alpha}^{\alpha} = \hbar\mu_1 \quad \hat{C}_2 = \hat{G}_{\alpha}^{\beta} \hat{G}_{\beta}^{\alpha} = \hbar^2\mu_2, \dots \quad (3)$$

is understood. The manifest expression of the structure constants  $f$  for the commutators

$$[\hat{L}_m^I, \hat{L}_n^J] = \hat{L}_m^I \hat{L}_n^J - \hat{L}_n^J \hat{L}_m^I = f^{IJK}[\vec{\mu}] \hat{L}_l^K \quad (4)$$

of a pair of operators (1) of  $\mathcal{L}_{\bar{\mu}}(u(N_+, N_-))$  entails an unpleasant and difficult computation, because of inherent ordering problems. However, the essence of the full quantum algebra  $\mathcal{L}_{\bar{\mu}}(u(N_+, N_-))$  can be still captured in a classical construction by extending the Poisson–Lie bracket

$$\{L_m^I, L_n^J\}_{\text{PL}} = (\eta_{\alpha_1\beta_2} G_{\alpha_2\beta_1} - \eta_{\alpha_2\beta_1} G_{\alpha_1\beta_2}) \frac{\partial L_m^I}{\partial G_{\alpha_1\beta_1}} \frac{\partial L_n^J}{\partial G_{\alpha_2\beta_2}} \quad (5)$$

of a pair of functions  $L_m^I, L_n^J$  on the commuting coordinates  $G_{\alpha\beta}$  to its deformed version, in the sense of [17]. To perform calculations with (5) is still rather complicated because of the non-canonical brackets for the generating elements  $G_{\alpha\beta}$ . Nevertheless, there is a standard boson operator realization  $G_{\alpha\beta} \equiv a_\alpha \bar{a}_\beta$  of the generators of  $u(N_+, N_-)$  for which things simplify greatly. Indeed, we shall understand that the quotient by the ideal generated by polynomials  $G_{\alpha_1\beta_1} G_{\alpha_2\beta_2} - G_{\alpha_1\beta_2} G_{\alpha_2\beta_1}$  is taken, so that the Poisson–Lie bracket (5) coincides with the standard Poisson bracket

$$\{L_m^I, L_n^J\}_{\text{P}} = \eta_{\alpha\beta} \left( \frac{\partial L_m^I}{\partial a_\alpha} \frac{\partial L_n^J}{\partial \bar{a}_\beta} - \frac{\partial L_m^I}{\partial \bar{a}_\beta} \frac{\partial L_n^J}{\partial a_\alpha} \right) \quad (6)$$

for the Heisenberg–Weyl algebra. There is basically only one possible deformation of the bracket (6)—corresponding to a normal ordering—that fulfils the Jacobi identities [17], which is the Moyal bracket [18]:

$$\{L_m^I, L_n^J\}_{\text{M}} = L_m^I * L_n^J - L_n^J * L_m^I = \sum_{r=0}^{\infty} 2 \frac{(\hbar/2)^{2r+1}}{(2r+1)!} P^{2r+1}(L_m^I, L_n^J) \quad (7)$$

where  $L * L' \equiv \exp(\frac{\hbar}{2}P)(L, L')$  is an invariant associative  $*$ -product and

$$P^r(L, L') \equiv \Upsilon_{i_1 j_1} \dots \Upsilon_{i_r j_r} \frac{\partial^r L}{\partial x_{i_1} \dots \partial x_{i_r}} \frac{\partial^r L'}{\partial x_{j_1} \dots \partial x_{j_r}} \quad (8)$$

with  $x \equiv (a, \bar{a})$  and  $\Upsilon_{2N \times 2N} \equiv \begin{pmatrix} 0 & \eta \\ -\eta & 0 \end{pmatrix}$ . We set  $P^0(L, L') \equiv LL'$ ; see also that  $P^1(L, L') = \{L, L'\}_{\text{P}}$ . Note the resemblance between the Moyal bracket (7) for *covariant symbols*  $L_m^I$  and the standard commutator (4) for operators  $\hat{L}_m^I$ . It is worthwhile mentioning that Moyal brackets were identified as the primary quantum deformation  $\mathcal{W}_\infty$  of the classical algebra  $w_\infty$  of area-preserving diffeomorphisms of the cylinder (see [19]).

With this information at hand, the manifest expression of the structure constants  $f$  for the Moyal bracket (7) is the following:

$$\begin{aligned} \{L_m^I, L_n^J\}_{\text{M}} &= \sum_{r=0}^{\infty} 2 \frac{(\hbar/2)^{2r+1}}{(2r+1)!} \eta^{\alpha_0\alpha_0} \dots \eta^{\alpha_{2r}\alpha_{2r}} f_{mn}^{IJ}(\alpha_0, \dots, \alpha_{2r}) L_{m+n}^{I+J-\sum_{j=0}^{2r} \delta_{\alpha_j}} \\ f_{mn}^{IJ}(\alpha_0, \dots, \alpha_{2r}) &= \sum_{\wp \in \Pi_2^{(2r+1)}} (-1)^{\ell_\wp+1} \prod_{s=0}^{2r} f_\wp(I_{\alpha_\wp^{(s)}}^{(s)}, m) f_\wp(J_{\alpha_\wp^{(s)}}^{(s)}, -n) \\ f_\wp(I_{\alpha_\wp^{(s)}}^{(s)}, m) &= I_{\alpha_\wp^{(s)}}^{(s)} + (-1)^{\theta(s-\ell_\wp)} \frac{1}{2} \left( \sum_{\beta > \alpha_\wp^{(s)}} m_{\alpha_\wp^{(s)}\beta} - \sum_{\beta < \alpha_\wp^{(s)}} m_{\beta\alpha_\wp^{(s)}} \right) \\ I_{\alpha_\wp^{(s)}}^{(s)} &= I_{\alpha_\wp^{(s)}} - \sum_{t=(\ell_\wp+1)\theta(s-\ell_\wp)}^{s-1} \delta_{\alpha_\wp^{(t)}, \alpha_\wp^{(s)}} \quad I^{(0)} = I^{(\ell_\wp+1)} \equiv I \\ \theta(s - \ell_\wp) &= \begin{cases} 0, & s \leq \ell_\wp \\ 1, & s > \ell_\wp \end{cases} \quad \delta_{\alpha_j} = (\delta_{1,\alpha_j}, \dots, \delta_{N,\alpha_j}) \end{aligned} \quad (9)$$

where  $\Pi_2^{(2r+1)}$  denotes the set of all possible partitions  $\wp$  of a string  $(\alpha_0, \dots, \alpha_{2r})$  of length  $2r + 1$  into two substrings

$$\overbrace{(\alpha_{\wp(0)}, \dots, \alpha_{\wp(\ell)})}^{\ell_{\wp}} \overbrace{(\alpha_{\wp(\ell+1)}, \dots, \alpha_{\wp(2r)})}^{2r+1-\ell_{\wp}} \quad (10)$$

of length  $\ell_{\wp}$  and  $2r + 1 - \ell_{\wp}$ , respectively. The number of elements  $\wp$  in  $\Pi_2^{(2r+1)}$  is clearly  $\dim(\Pi_2^{(2r+1)}) = \sum_{\ell=0}^{2r+1} \frac{(2r+1)!}{(2r+1-\ell)!\ell!} = 2^{2r+1}$ .

For  $r = 0$ , there are just two partitions:  $(\alpha)(\cdot)$ ,  $(\cdot)(\alpha)$ , and the leading (classical,  $\hbar \rightarrow 0$ ) structure constants are, for example:

$$f_{mn}^{IJ}(\alpha) = J_{\alpha} \left( \sum_{\beta>\alpha} m_{\alpha\beta} - \sum_{\beta<\alpha} m_{\beta\alpha} \right) - I_{\alpha} \left( \sum_{\beta>\alpha} n_{\alpha\beta} - \sum_{\beta<\alpha} n_{\beta\alpha} \right). \quad (11)$$

In this limit they reproduce the Virasoro commutation relations for the particular generators  $V_k^{(\alpha\beta)} \equiv L_{ke_{\alpha\beta}}^{\delta_{\alpha}}$ , where  $k \in \mathbb{Z}$  and  $e_{\alpha\beta}$  denotes an upper-triangular matrix with zero entries except for one at the  $(\alpha\beta)$ -position. Indeed, there are  $N(N-1)$  non-commuting Virasoro sectors in (9), corresponding to each root in  $SU(N_+, N_-)$ , with classical commutation relations:

$$\{V_k^{(\alpha\beta)}, V_l^{(\alpha\beta)}\}_{\text{P}} = \eta^{\alpha\alpha} \text{sign}(\beta - \alpha)(k - l)V_{k+l}^{(\alpha\beta)}. \quad (12)$$

For large  $r$ , we can benefit from the use of algebraic-computing programs such as [20] to deal with the high number of partitions.

Our boson operator realization  $G_{\alpha\beta} \equiv a_{\alpha}\bar{a}_{\beta}$  of the  $u(N_+, N_-)$  generators corresponds to the particular case of  $\vec{\mu}_0 = (N, 0, \dots, 0)$  for the Casimir eigenvalues, so that the commutation relations (9) are related to the particular algebra  $\mathcal{L}_{\vec{\mu}_0}^{\Lambda}(u(N_+, N_-))$  (see below for more general cases). A different (minimal) realization of  $\mathcal{L}_{\mu}(su(1, 1))$  in terms of a single boson  $(a, \bar{a})$ , which corresponds to  $\mu_c = \bar{s}_c(\bar{s}_c - 1) = -\frac{3}{16}$  for the critical value  $\bar{s}_c = \frac{3}{4}$  of the *symplin* degree of freedom  $\bar{s}$ , was given in [21]; this case is also related to the *symplecton* algebra of [14]. Note the close resemblance between the algebra (9)—and the leading structure constants (11)—and the quantum deformation  $\mathcal{W}_{\infty} \simeq \mathcal{L}_0(su(1, 1))$  of the algebra of area-preserving diffeomorphisms of the cylinder [8, 19], although we recognize that the case discussed in this letter is far richer.

If the analyticity of the symbols  $L_m^I$  of (1) is taken into account, then one should worry about a restriction of the range of the indices  $I_{\alpha}, m_{\alpha\beta}$ . The subalgebra  $\mathcal{L}_{\vec{\mu}_0}^{\Lambda}(u(N_+, N_-)) \equiv \{L_m^I | \Lambda_{\alpha} = I_{\alpha} - (\sum_{\beta>\alpha} |m_{\alpha\beta}| + \sum_{\beta<\alpha} |m_{\beta\alpha}|)/2 \in \mathbb{N}\}^{\dagger}$  of polynomial functions on  $G_{\alpha\beta}$ , the structure constants  $f_{mn}^{IJ}(\alpha_0, \dots, \alpha_{2r})$  of which are zero for  $r > (\sum_{\alpha} (I_{\alpha} + J_{\alpha}) - 1)/2$ , can be extended beyond the ‘wedge’  $\Lambda \geq 0$  by analytic continuation, that is, by revoking this restriction to  $\Lambda \in \mathbb{Z}/2$ . The aforementioned ‘extension beyond the wedge’ (see [8, 15] for similar concepts) makes possible the existence of conjugated pairs  $(L_m^I, L_{-m}^{I'})$ , with  $\sum_{\alpha} I_{\alpha} + I'_{\alpha} = 2r + 1$  and  $I_{\alpha} + I'_{\alpha} \equiv r_{\alpha} \in \mathbb{N}$ , that give rise to central terms under commutation:

$$\Xi(L_m^I, L_n^{I'}) = \frac{\hbar^{2r+1} 2^{-2r} (-1)^{\sum_{\alpha=N+1}^{r_{\alpha}} r_{\alpha}}}{\prod_{\alpha=1}^N (2r + 1 - r_{\alpha})!} f_{m,-m}^{II'}(1^{(r_1)}, \dots, N^{(r_N)}) \delta_{m+n,0} \hat{1} \quad (13)$$

where  $(1^{(r_1)}, \dots, N^{(r_N)})$  is a string of length  $\sum_{\alpha=1}^N r_{\alpha} = 2r + 1$ , and  $\alpha^{(r_{\alpha})}$  denotes a substring made of  $r_{\alpha}$ -times  $\alpha$ , for each  $\alpha$ . The generator  $\hat{1} \equiv L_0^0$  is central (it commutes with everything) and the Lie algebra two-cocycle (13) defines a non-trivial central extension of  $\mathcal{L}_{\vec{\mu}_0}^{\Lambda}(u(N_+, N_-))$  by  $U(1)$ .

A thorough study of the Lie-algebra cohomology of  $\mathcal{L}_{\vec{\mu}}(u(N_+, N_-))$  and its irreps still remains to be accomplished; it requires separate attention and shall be left for future works.

$\dagger$   $\mathbb{N}, \mathbb{Z}, \mathbb{R}$  and  $\mathbb{C}$  denote the set of natural, integer, real and complex numbers, respectively.

Two-cocycles such as (13) provide the essential ingredient to construct invariant geometric action functionals on coadjoint orbits of  $\mathcal{L}_{\bar{\mu}}(u(N_+, N_-))$ —see, e.g., [10] for the derivation of the WZNW action of  $D = 2$  matter fields coupled to a chiral  $\mathcal{W}_\infty$  gravity background from  $\mathcal{W}_\infty \simeq \mathcal{L}_0(su(1, 1))$ .

In order to deduce the structure constants for general  $\mathcal{L}_{\bar{\mu}}(u(N_+, N_-))$  from  $\mathcal{L}_{\bar{\mu}_0}(u(N_+, N_-))$ , a procedure similar to that of [12], for the particular case of  $\mathcal{U}(sl(2, \mathbb{R}))$ , can be applied. Special attention must be paid to the limit  $\lim_{\hbar \rightarrow 0} \mathcal{L}_{\bar{\mu}}(u(N_+, N_-)) \simeq \mathcal{P}_{\mathbb{C}}(M_{N_+, N_-})$ , which coincides with the Poisson algebra of complex (wave)functions  $\psi_{|m|}^I, \psi_{-|m|}^I \equiv \bar{\psi}_{|m|}^I$  on algebraic manifolds (coadjoint orbits [22])  $M_{N_+, N_-} \simeq U(N_+, N_-)/U(1)^{N \dagger}$ . It is well known that there exists a natural symplectic structure  $(M_{N_+, N_-}, \Omega)$ , which defines the Poisson algebra

$$\{\psi_m^I, \psi_n^J\} = \Omega^{\alpha_1 \beta_1; \alpha_2 \beta_2} \frac{\partial \psi_m^I}{\partial g^{\alpha_1 \beta_1}} \frac{\partial \psi_n^J}{\partial g^{\alpha_2 \beta_2}} \quad (14)$$

and an invariant symmetric bilinear form  $\langle \psi_m^I | \psi_n^J \rangle = \int v(g) \bar{\psi}_m^I(g) \psi_n^J(g)$  given by the natural invariant measure  $v(g) \sim \Omega^{N(N-1)/2}$  on  $U(N_+, N_-)$ , where  $g^{\alpha\beta} = \bar{g}^{\beta\alpha} \in \mathbb{C}, \alpha \neq \beta$ , is a (local) system of complex coordinates on  $M_{N_+, N_-}$ . The structure constants for (14) can be obtained through  $f_{mnk}^{IJL} = \langle \psi_k^L | \{\psi_m^I, \psi_n^J\} \rangle$ . Also, an associative  $\star$ -product can be defined through the convolution of two functions  $(\psi_m^I \star \psi_n^J)(g') \equiv \int v(g) \psi_m^I(g) \psi_n^J(g^{-1} \bullet g')$ , which gives the algebra  $\mathcal{P}_{\mathbb{C}}(M_{N_+, N_-})$  a non-commutative character— $g \bullet g'$  denotes the group composition law of  $U(N_+, N_-)$ . The derivation of a manifest expression for all these structures is still in progress [23].

Taking advantage of all these geometrical tools, action functionals for  $\mathcal{L}_\infty(u(N_+, N_-))$  Yang–Mills gauge theories in  $D$  dimensions could be built as

$$\begin{aligned} S &= \int d^D x \langle F_{\nu\gamma}(x, g) | F^{\nu\gamma}(x, g) \rangle \\ F_{\nu\gamma} &= \partial_\nu A_\gamma - \partial_\gamma A_\nu + \{A_\nu, A_\gamma\} \\ A_\nu(x, g) &= A_{\nu I}^m(x) \psi_m^I(g) \quad \nu, \gamma = 1, \dots, D \end{aligned} \quad (15)$$

the ‘vacuum configurations’ (spacetime-constant potentials  $X_\nu(g) \equiv A_\nu(0, g)$ ) of which, define the action for higher-extended objects:  $N(N - 1)$ -‘branes’, in the usual nomenclature. Here,  $\mathcal{L}_\infty(u(N_+, N_-))$  plays the role of gauge symplectic (volume-preserving) diffeomorphisms  $L_\psi \equiv \{\psi, \cdot\}$  on the  $N(N - 1)$ -brane  $M_{N_+, N_-}$ . A particularly interesting case might be  $SU(2, 2) = U(2, 2)/U(1)$ : the conformal group in  $3 + 1$  (or the AdS group in  $4 + 1$ ) dimensions, in an attempt to construct ‘conformal gravities’ in realistic dimensions. The infinite-dimensional algebra  $\mathcal{L}_\mu(u(2, 2))$  might be seen as the *generalization of the Virasoro (two-dimensional) conformal symmetry to  $3 + 1$  dimensions*.

Finally, let me comment on the potential relevance of the  $C^*$ -algebras  $\mathcal{L}_{\bar{\mu}}(\mathcal{G})$  on tractable non-commutative versions [24] of symmetric curved spaces  $M = G/H$ , where the notion of a pure state  $\psi_m^I$  replaces that of a point. The possibility of describing phase-space physics in terms of the quantum analogue of the algebra of functions (the covariant symbols  $L_m^I$ ), and the absence of localization expressed by the Heisenberg uncertainty relation, was noticed a long time ago by Dirac [25]. Just as the standard differential geometry of  $M$  can be described by using the algebra  $C^\infty(M)$  of smooth complex functions  $\psi$  on  $M$  (that is,  $\lim_{\hbar \rightarrow 0} \mathcal{L}_{\bar{\mu}}(\mathcal{G})$ , when considered as an associative, commutative algebra), a non-commutative geometry for  $M$  can be described by using the algebra  $\mathcal{L}_{\bar{\mu}}(\mathcal{G})$ , seen as an associative algebra with a non-commutative  $\star$ -product like (7), (8). The appealing feature of a non-commutative space  $M$  is that a  $G$ -invariant ‘lattice structure’ can be constructed in a natural way, a desirable property as regards

$\dagger$  For  $N_- \neq 0$ , other cases could be also contemplated (e.g. continuous series of  $SU(1, 1)$ ).

finite models of quantum gravity (see e.g., [26] and references therein). Indeed, as already mentioned,  $\mathcal{L}_{\bar{\mu}}(\mathcal{G})$  collapses to  $\text{Mat}_d(\mathbb{C})$  (the full matrix algebra of  $d \times d$  complex matrices) whenever  $\mu_\alpha$  coincides with the eigenvalue of  $\hat{C}_\alpha$  in a  $d$ -dimensional irrep  $D_{\bar{\mu}}$  of  $G$ . This fact provides a finite ( $d$ -points) ‘fuzzy’ or ‘cellular’ description of the non-commutative space  $M$ , the classical (commutative) case being recovered in the limit  $\bar{\mu} \rightarrow \infty$ . The notion of space itself could be the collection of all of them, enclosed in a single irrep of  $\mathcal{L}_{\bar{\mu}}(\mathcal{G})$  for general  $\bar{\mu}$ , with different multiplicities, as it actually happens with the reduction of an irrep of the centrally extended Virasoro group under its  $SL(2, \mathbb{R})$  subgroup [27]; the multiplicity should increase with  $\bar{\mu}$  (‘the density of points’), so that classical-like spaces are more abundant. It is also a very important feature of  $\mathcal{L}_{\bar{\mu}}(\mathfrak{u}(N_+, N_-))$  that the quantization deformation scheme (7) does not affect the maximal finite-dimensional subalgebra  $\mathfrak{su}(N_+, N_-)$  (‘good observables’ or preferred coordinates [17]) of non-commuting ‘position operators’

$$\begin{aligned} y_{\alpha\beta} &= \frac{\lambda}{2\hbar} (\hat{G}_{\alpha\beta} + \hat{G}_{\beta\alpha}) & y_{\beta\alpha} &= \frac{i\lambda}{2\hbar} (\hat{G}_{\alpha\beta} - \hat{G}_{\beta\alpha}) & \alpha < \beta \\ y_\alpha &= \frac{\lambda}{\hbar} (\eta_{\alpha\alpha} \hat{G}_{\alpha\alpha} - \eta_{\alpha+1, \alpha+1} \hat{G}_{\alpha+1, \alpha+1}) \end{aligned} \quad (16)$$

on the algebraic manifold  $M_{N_+, N_-}$ , where  $\lambda$  denotes a parameter that gives  $y$  dimensions of length (e.g., the square root of the Planck area  $\hbar G$ ). The ‘volume’  $v_j$  of the  $N - 1$  submanifolds  $M_j$  of the flag manifold  $M_{N_+, N_-} = M_N \supset \dots \supset M_2$  (see, e.g., [28] for a definition of flag manifolds) is proportional to the eigenvalue  $\mu_j$  of the  $\mathfrak{su}(N_+, N_-)$  Casimir operator  $\hat{C}_j$  in those coordinates:  $v_j = \lambda^j \mu_j$ . Large volumes (flat-like spaces) correspond to a high density of points (large  $\mu$ ). In the classical limit  $\lambda \rightarrow 0$ ,  $\mu \rightarrow \infty$ , the  $y$  coordinates commute.

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